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The non-linear Saint-Venant problem of the torsion, stretching and bending of a naturally twisted rod☆

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Abstract

Problems on large stretching, torsional and bending deformations of a naturally twisted rod, loaded with end forces and moments, are considered from the point of view of the non-linear three-dimensional theory of elasticity. Particular solutions of the equations of elastostatics are found, which are two-parameter families of finite deformations and which possess the property that, for these deformations, the initial system of three-dimensional non-linear equations reduces to a system of equations with two independent variables. The use of these equations enables one to reduce certain Saint-Venant problems for a naturally twisted rod to two-dimensional non-linear boundary-value problems for a planar domain in the form of the cross-section of a rod. Different formulations of the two-dimensional boundary-value problem for the cross-section are proposed, which differ in the choice of the unknown functions. A non-linear problem of the torsion and stretching of a circular cylinder with helical anisotropy, which is reduced to ordinary differential equations, is considered as a special case.

Saint-Venant problems for a naturally twisted rod have been considered in a number of publications[1–4] within the framework of the linear theory of elasticity.

1. Transformation to a two-dimensional boundary-value problem

Consider an elastic body which, in the reference configuration, has the form of a naturally twisted rod with a rectilinear axis. The body has been formed by a helical motion along the x_3 axis of a planar figure σ which is located in a plane perpendicular to this axis. In describing the deformation of the elastic medium, we shall use non-orthogonal curvilinear coordinates[1] y_1, y_2, y_3, which are connected with the Cartesian coordinates of the reference configuration x_1, x_2, x_3 by the relations

$$x_1 = y_1\cos\alpha x_3 - y_2\sin\alpha x_3, \quad x_2 = y_1\sin\alpha x_3 + y_2\cos\alpha x_3, \quad x_3 = y_3; \quad \alpha = \text{const} \tag{1.1}$$

as Lagrange coordinates. Here α is the natural twist angle, and y_1 and y_2 are Cartesian coordinates in the plane of the domain σ. We shall write the equation of the piecewise-smooth contour which bounds the domain σ in parametric form: $y_1 = y_1(t)$, $y_2 = y_2(t)$ and we shall refer to the helical surface formed by the helical motion of the curve $\partial\sigma$ along the x_3 axis as the lateral surface of the rod. Assuming that the parameters t and y_3 are Gaussian coordinates, we will

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write the equation for the lateral surface of the body in the form

$$\mathbf{r}(t, y_3) = y_1(t)\mathbf{d}_1 + y_2(t)\mathbf{d}_2 + y_3\mathbf{i}_3 \tag{1.2}$$

$$\mathbf{d}_1 = \mathbf{i}_1\cos\alpha y_3 + \mathbf{i}_2\sin\alpha y_3, \quad \mathbf{d}_2 = -\mathbf{i}_1\sin\alpha y_3 + \mathbf{i}_2\cos\alpha y_3 \tag{1.3}$$

where $\mathbf{r} = x_m\mathbf{i}_m$ is the radius vector of a point of the surface and $\mathbf{i}_m (m = 1, 2, 3)$ are the constant unit vectors of the Cartesian coordinates. The unit vector of the normal to the lateral surface

$$\mathbf{n} = \frac{\dot{y}_2\mathbf{d}_1 - \dot{y}_1\mathbf{d}_2 + \alpha(y_1\dot{y}_1 + y_2\dot{y}_2)\mathbf{i}_3}{\sqrt{\dot{y}_1^2 + \dot{y}_2^2 + \alpha^2(y_1\dot{y}_1 + y_2\dot{y}_2)^2}} = n_s\mathbf{d}_s, \quad s = 1, 2, 3; \quad \mathbf{d}_3 = \mathbf{i}_3 \tag{1.4}$$

is found using relations (1.2). A derivative with respect to t is denoted by a dot. The equality

$$n_3 = \alpha(y_2n_1 - y_1n_2) \tag{1.5}$$

follows from expression (1.4), and the vector $n_1\mathbf{d}_1+n_2\mathbf{d}_2$ is normal to the plane of the curve $\partial\sigma$. The Cartesian coordinates of the points of the deformed body (the Euler coordinates) are denoted by $X_k (k = 1, 2, 3)$, and we will consider the following two-parameter family of deformations of the naturally twisted beam

$$\begin{aligned} X_1 &= u_1(y_1, y_2)\cos\psi y_3 - u_2(y_1, y_2)\sin\psi y_3 \\ X_2 &= u_1(y_1, y_2)\sin\psi y_3 + u_2(y_1, y_2)\cos\psi y_3 \\ X_3 &= \lambda y_3 + u_3(y_1, y_2); \quad \lambda, \psi = \text{const} \end{aligned} \tag{1.6}$$

The radius vector of a point of the deformed body $\mathbf{R} = X_k\mathbf{i}_k$ is represented in the form

$$\begin{aligned} &\mathbf{R}(y_1, y_2, y_3) = u_k\mathbf{h}_k + \lambda y_3\mathbf{h}_3 \\ &\mathbf{h}_1 = \mathbf{i}_1\cos\psi y_3 + \mathbf{i}_2\sin\psi y_3, \quad \mathbf{h}_2 = -\mathbf{i}_1\sin\psi y_3 + \mathbf{i}_2\cos\psi y_3, \quad \mathbf{h}_3 = \mathbf{i}_3 \end{aligned} \tag{1.7}$$

Using representation (1.7) for the gradient of the deformation $\mathbf{C} = \operatorname{grad}\mathbf{R}$, where grad is the gradient operator in Lagrange coordinates, we obtain ($u_{k,p} = \partial u_k/\partial y_p$)

$$\begin{aligned} \mathbf{C}(y_1, y_2, y_3) &= (\mathbf{d}_1 + \alpha y_2\mathbf{i}_3)\otimes\partial\mathbf{R}/\partial y_1 + (\mathbf{d}_2 - \alpha y_1\mathbf{i}_3)\otimes\partial\mathbf{R}/\partial y_2 + \\ &+ \mathbf{i}_3\otimes\partial\mathbf{R}/\partial y_3 = C_{sk}(y_1, y_2)\mathbf{d}_s\otimes\mathbf{h}_k \end{aligned} \tag{1.8}$$

$$\begin{aligned} &C_{pk} = u_{k,p}, \quad C_{31} = \tilde{C}_{31} - \psi u_2, \quad C_{32} = \tilde{C}_{32} + \psi u_1, \quad C_{33} = \tilde{C}_{33} + \lambda \\ &\tilde{C}_{3k} \equiv -\alpha y_1u_{k,2} + \alpha y_2u_{k,1}; \quad p = 1, 2; \quad k = 1, 2, 3 \end{aligned} \tag{1.9}$$

A measure of the Cauchy deformation

$$\mathbf{G} = \mathbf{C}\cdot\mathbf{C}^T = G_{sk}(y_1, y_2)\mathbf{d}_s\otimes\mathbf{d}_k; \quad G_{sk} = C_{sm}C_{km} \tag{1.10}$$

is determined from relations (1.9), that is, the components of the tensor $\mathbf{G}$ in the orthonormalized basis $\mathbf{d}_s$ are independent of the coordinate y_3.

The geometric meaning of representations (1.6) lies in the fact that a cross-section of the rod, which is a distance y_3 from the origin of the coordinates, undergoes a certain planar deformation which is specified by the functions u_1 and u_2, a finite rotation about the x_3 axis by an angle $(\psi - \alpha)y_3$, a translational displacement along the axis by an amount $(\lambda - 1)y_3$ and warping, which is described by the function u_3. The case when $\psi = 0$ corresponds to a straightening deformation of the naturally twisted rod during which the rod is converted into a prismatic beam. Neglecting mass forces, we write the equations for the statics of an elastic body using the non-symmetric Piola stress tensor $\mathbf{D}$[5]

$$\operatorname{div}\mathbf{D} = 0 \tag{1.11}$$

$$\mathbf{D} = dW/d\mathbf{C} = \mathbf{P}\cdot\mathbf{C}, \quad \mathbf{P} = 2dW/d\mathbf{G} \tag{1.12}$$

Here, div is the divergence operator in Lagrange coordinates, **P** is the symmetric Kirchhoff tensor and $W(\mathbf{G})$ is the specific potential energy of deformation.

It is henceforth assumed that the specific energy of the elastic material W, which is considered as a function of the components G_{sk} of the Cauchy measure of deformation in the basis $\mathbf{d}_s$, is explicitly independent of the coordinate $y_3 = x_3$ but can depend on the coordinates y_1, y_2: $W = W(G_{sk}, y_1, y_2)$. We shall say that such materials are homogeneous with respect to the y_3 coordinate. The above-mentioned class of materials includes isotropic elastic media with an arbitrary inhomogeneity along the y_1 and y_2 coordinates, which are measured in the plane of the cross-section σ and, also, several kinds of anisotropic media.

Since the quantities G_{sk} are independent of the y_3 coordinate, it follows from relations (1.12) that, in the case of a material which is homogeneous with respect to the y_3 coordinate, the components $P_{sk} = \mathbf{d}_s \cdot \mathbf{P} \cdot \mathbf{d}_k$ of the Kirchhoff stress tensor will be functions solely of the coordinates y_1 and y_2. On the basis of relations (1.8), the Piola stress tensor for a deformation of the form (1.6) will therefore have the representation

$$\mathbf{D}(y_1, y_2, y_3) = D_{sk}(y_1, y_2)\mathbf{d}_s \otimes \mathbf{h}_k \tag{1.13}$$

Substituting expression (1.13) into equality (1.11), we obtain the scalar form of the equilibrium equation for the Piola stresses

$$\begin{gathered} \tilde{D}_1 - \psi D_{32} = 0, \quad \tilde{D}_2 + \psi D_{31} = 0, \quad \tilde{D}_3 = 0 \\ \tilde{D}_k \equiv D_{1k,1} + D_{2k,2} - \alpha y_1 D_{3k,2} + \alpha y_2 D_{3k,1}, \quad k = 1, 2, 3 \end{gathered} \tag{1.14}$$

Taking account of the equation of state (1.12) and relations (1.8)–(1.10), (1.13), we see that Eqs. (1.14) are a system of three scalar equations in three functions of two variables $u_k(y_1, y_2)$ $(k = 1, 2, 3)$.

The scalar form of the boundary conditions $\mathbf{n}\,\mathbf{D} = 0$ on the lateral surface of the rod which is assumed to be free, load- in accordance with relations (1.4), (1.5) and (1.13), is as follows:

$$n_1(D_{1k} + \alpha y_2 D_{3k}) + n_2(D_{2k} - \alpha y_1 D_{3k}) = 0, \quad k = 1, 2, 3 \tag{1.15}$$

Since, according to expression (1.4), the vector components of the normals n_1 and n_2 are independent of the y_3 coordinate, the boundary conditions (1.15) do not contain the y_3 variable and, together with the equilibrium Eq. (1.14), form a two-dimensional boundary-value problem for the plane of the domain σ with the unknown functions $u_k(y_1, y_2)$. The constants ψ and λ are assumed to be specified parameters.

Hence, the assumptions (1.6) concerning the nature of the deformation of an elastic medium reduce the initial non-linear three-dimensional problem for a naturally twisted rod to a two-dimensional boundary-value problem for the planar domain σ in the form of the cross-section of the rod.

Suppose $u_k(y_1, y_2)$ is a certain solution of boundary-value problem (1.14), (1.15). Using relations (1.8)–(1.10), (1.12) and (1.13), it can be verified that the functions

$$u_1^* = u_1\cos\kappa - u_2\sin\kappa, \quad u_2^* = u_1\sin\kappa + u_2\cos\kappa, \quad u_3^* = u_3 + \gamma \tag{1.16}$$

where κ and γ are arbitrary real constants, also satisfy Eqs. (1.14) and conditions (1.15).

The insensitivity of the boundary-value problem in the cross-section to the substitution (1.16) means that the position of the elastic solid after deformation is defined, apart from a rotation about the X_3 axis and translational displacement along the same axis. This non-uniqueness of the solution can be avoided by subjecting the unknown functions to additional conditions. The integral relations

$$\iint u_3 d\sigma = 0, \quad \iint(\cos\beta - 1)d\sigma = 0 \tag{1.17}$$

where

$$\cos\beta = \frac{u_{1,1} + u_{2,2}}{\sqrt{(u_{1,1} + u_{2,2})^2 + (u_{2,1} - u_{1,2})^2}} \tag{1.18}$$

are one of the versions of these conditions. Here and henceforth a double integral is taken over the domain σ.

The quantity β, defined by formula (1.18), is known to be [Ref. 6, p. 91] the angle of rotation of the material fibres accompanying a finite planar deformation. The second constraint (1.17) therefore implies that, on average, there is no rotation of the particles of the rod about its axis through the cross-section $y_3 = 0$. The first constraint of (1.17) implies that, on average, the axial displacement of the particles of the rod when $y_3 = 0$ is zero throughout the cross-section.

When account is taken of the constraints (1.17), we would expect the solution of problem (1.14), (1.15) to be unique. In fact, non-uniqueness of the solution would imply the existence of some forms of loss of stability of the rod for which the deformation is the same for all cross-sections. If this type of equilibrium bifurcation is also possible, then it is for very large values of the parameters ψ and $\lambda - 1$.

We will now consider another two-parameter family of deformations of a naturally twisted beam, which is analogous to a deformation involving the three-dimensional bending of a prismatic body[7]

$$\begin{aligned} X_1 &= \upsilon_1(y_1, y_2) + l y_3 \\ X_2 &= \upsilon_2(y_1, y_2)\cos\omega y_3 - \upsilon_3(y_1, y_2)\sin\omega y_3 \\ X_3 &= \upsilon_2(y_1, y_2)\sin\omega y_3 + \upsilon_3(y_1, y_2)\cos\omega y_3; \quad l, \omega = \text{const} \end{aligned} \tag{1.19}$$

In this case, the helices $y_1 = \text{const}$ and $y_2 = \text{const}$, the axes of which are parallel to the unit vector $\mathbf{i}_3$, are converted after deformation into helices, the axes of which are parallel to the unit vector $\mathbf{i}_1$ and the lateral surface of the beam is transformed into a helical surface, the axis of which coincides with the X_1 axis. When $l = 0$, the helical lateral surface of the beam is transformed after deformation into a sector of a surface of revolution, that is, the naturally twisted rod is transformed into a curved beam with a circular axis located in the X_2X_3 plane. According to equalities (1.19), the radius vector of a point of the deformed body has the form

$$\begin{aligned} &\mathbf{R} = \upsilon_k(y_1, y_2)\mathbf{g}_k + l y_3 \mathbf{g}_1, \quad k = 1, 2, 3 \\ &\mathbf{g}_1 = \mathbf{i}_1, \quad \mathbf{g}_2 = \mathbf{i}_2\cos\omega y_3 + \mathbf{i}_3\sin\omega y_3, \quad \mathbf{g}_3 = -\mathbf{i}_2\sin\omega y_3 + \mathbf{i}_3\cos\omega y_3 \end{aligned} \tag{1.20}$$

and, for the gradient of the deformation and the measure of the Cauchy deformation, we obtain

$$\mathbf{C} = C_{mn}(y_1, y_2)\mathbf{d}_m \otimes \mathbf{g}_n, \quad m, n = 1, 2, 3 \tag{1.21}$$

$$\begin{aligned} &C_{pn} = \upsilon_{n,p}, \quad C_{31} = \tilde{C}_{31} + l, \quad C_{32} = \tilde{C}_{32} - \omega\upsilon_3, \quad C_{33} = \tilde{C}_{33} + \omega\upsilon_2; \quad p = 1, 2 \\ &\tilde{C}_{3n} \equiv \alpha y_2 \upsilon_{n,1} - \alpha y_1 \upsilon_{n,2} \end{aligned} \tag{1.22}$$

$$\mathbf{G} = C_{ml}C_{nl}\mathbf{d}_m \otimes \mathbf{d}_n \tag{1.23}$$

The Piola stress tensor

$$\mathbf{D}(y_1, y_2, y_3) = D_{mn}(y_1, y_2)\mathbf{d}_m \otimes \mathbf{g}_n \tag{1.24}$$

follows from this in the case of an elastic medium which is homogeneous about the y_3 coordinate.

By virtue of relations (1.11), (1.20) and (1.24) and the last equality of (1.14), the equilibrium equations with respect to the stresses in the three-dimensional bending problem are written in the form

$$\tilde{D}_1 = 0, \quad \tilde{D}_2 - \omega D_{33} = 0, \quad \tilde{D}_3 - \omega D_{32} = 0 \tag{1.25}$$

The boundary conditions for the stresses on the lateral surface of the rod in the case of bending are similar in form to conditions (1.15). The two-dimensional boundary-value problem in the functions $\upsilon_n(y_1, y_2, y_3)$ ($n = 1, 2, 3$) is insensitive to the substitution

$$\upsilon_1^* = \upsilon_1 + \eta, \quad \upsilon_2^* = \upsilon_2\cos\theta - \upsilon_3\sin\theta, \quad \upsilon_3^* = \upsilon_2\sin\theta + \upsilon_3\cos\theta \tag{1.26}$$

where η and θ are arbitrary real constants. The non-uniqueness of the solution of the form (1.26), associated with the possibility of an arbitrary rotation of the body about the X_1 axis and arbitrary translational displacement along this axis, is removed after the imposition of conditions analogous to (1.17) and (1.18),

$$\iint(\upsilon_1 - y_1)d\sigma = 0, \quad \iint\left(\frac{\upsilon_{2,1} + \upsilon_{3,2}}{\sqrt{(\upsilon_{2,1} + \upsilon_{3,2})^2 + (\upsilon_{3,1} - \upsilon_{2,2})^2}} - 1\right)d\sigma = 0 \tag{1.27}$$

2. Forces and moments acting on the ends of the beam

The families of finite deformations (1.6) and (1.19) represent an interchange in the equations of non-linear elastostatics which leads to a theory of the torsion and bending of a naturally twisted rod which has the same accuracy as that which is inherent in the classical Saint-Venant problem[6] concerning the bending and torsion of a prismatic, linearly elastic solid, that is, the equilibrium equations in the bulk of the solid and the boundary conditions on the lateral surface are satisfied exactly, while the boundary conditions on the ends of the beam are satisfied approximately in the integral Saint-Venant sense.

According to relation (1.13), the principal vector of the forces which act in an arbitrary cross-section of the rod $y_3 = \text{const}$ accompanying deformation (1.6) has the form

$$\mathbf{F}(x_3) = \iint \mathbf{d}_3 \cdot \mathbf{D} d\sigma = F_1\mathbf{h}_1 + F_2\mathbf{h}_2 + F_3\mathbf{i}_3; \quad F_k = \iint D_{3k} d\sigma = \text{const} \tag{2.1}$$

Since the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ depend on y_3, it follows from expression (2.1) and the condition for the equilibrium of the part of the rod included between the two planes $y_3 = a$ and $y_3 = b$, where a and b are arbitrary numbers, that $F_1 = F_2 = 0$. Consequently, the principal vector of the forces in a cross-section of the beam is the same, in the case of torsional and stretching compression deformation (1.6), for all cross-sections and has the direction of the unit vector $\mathbf{i}_3$.

We shall calculate the principal moment $\mathbf{M}$ of the forces in a cross-section $y_3 = \text{const}$ with respect to a certain point in the line $X_1 = X_2 = 0$. According to expressions (1.6), this line is the axis of the helical surface into which the lateral surface of the rod is converted after deformation. Since the principal vector is parallel to the above mentioned line, the moment is independent of the datum point on the X_3 axis, which enables one to calculate the moment about the point $X_1 = X_2 = X_3 = 0$. On taking account of the fact that $F_1 = F_2 = 0$, we find, using relations (1.7) and (1.13), that

$$\mathbf{M}(x_3) = -\iint \mathbf{d}_3 \cdot \mathbf{D} \times \mathbf{R} d\sigma = M_k\mathbf{h}_k \tag{2.2}$$

$$M_1 = \iint (D_{33}u_2 - D_{32}u_3)d\sigma, \ M_2 = \iint (D_{31}u_3 - D_{33}u_1)d\sigma, \ M_3 = \iint (D_{32}u_1 - D_{31}u_2)d\sigma \tag{2.3}$$

It follows from these considerations and the conditions for the balance of the moments of all the forces applied to the segment of the rod $a \le y_3 \le b$ that $M_1 = M_2 = 0$.

Thus, it has been proved that the realization of the deformation (1.6) requires the application of a system of forces to the ends of a naturally twisted beam, which is statically equivalent to a longitudinal force F_3 acting at a point of the X_3 axis and a torque M_3. Henceforth, when considering the problem of stretching and torsion, we shall assume that the cross-section of the beam σ possesses central symmetry, that is, it is superimposed on itself after a rotation of 180° about the axis of the rod. A cross-section having the shape of the letter Z serves as an example. Cross-sections having two axes of symmetry clearly also belong to this class. If the y_3 axis passes through the centres of cross-sections with central symmetry, then the coordinate transformation which leaves the domain σ unchanged has the form

$$y_1' = -y_1, \quad y_2' = -y_2, \quad y_3' = y_3$$

We now consider the two-dimensional boundary-value problem which describes the torsion and stretching compression of a naturally twisted rod and consists of Eqs. (1.14), (1.17) and (1.18) and the boundary conditions (1.15). We make the following replacement of the independent variables and the unknown functions.

$$y_1' = -y_1, \quad y_2' = -y_2, \quad y_3' = y_3; \quad u_1' = -u_1, \quad u_2' = -u_2, \quad u_3' = u_3 \tag{2.4}$$

Theorem. *In the case of a homogeneous, isotropic, elastic solid, the boundary-value problem* (1.14), (1.15), (1.17), (1.18) *in the domain σ, which possesses central symmetry, is invariant under the transformation* (2.4).

Proof. According to relations (1.9) and (1.10), transformation (2.4) generates the following transformation of the components of the gradient of the deformation and the measure of Cauchy deformation

$$\begin{aligned} C_{pq}' &= C_{pq}, \quad C_{p3}' = -C_{p3}, \quad C_{3p}' = -C_{3p}, \quad C_{33}' = C_{33} \\ G_{pq}' &= G_{pq}, \quad G_{p3}' = -G_{p3}, \quad G_{33}' = G_{33}; \quad p, q = 1, 2 \end{aligned} \tag{2.5}$$

On the basis of formulae (2.5), the invariants of the measure of Cauchy deformation

$$I_1 = \mathrm{tr}\mathbf{G}, \quad I_2 = \frac{1}{2}(\mathrm{tr}^2\mathbf{G} - \mathrm{tr}\mathbf{G}^2), \quad I_3 = \det\mathbf{G}$$

are transformed as follows:

$$I'_k = I_k, \quad k = 1, 2, 3 \tag{2.6}$$

In the case of an isotropic, elastic material, the Kirchhoff stress tensor **P**, being an isotropic function of the tensor **G**, can be represented in the form[5] (**E** is the unit tensor)

$$\mathbf{P} = a_0(I_1, I_2, I_3)\mathbf{E} + a_1(I_1, I_2, I_3)\mathbf{G} + a_2(I_1, I_2, I_3)\mathbf{G}^2 \tag{2.7}$$

The rule for transforming the components of the Piola stress tensor in expansion (1.13) follows from relations (2.5)–(2.7)

$$D'_{pq} = D_{pq}, \quad D'_{p3} = -D_{p3}, \quad D'_{3p} = -D_{3p}, \quad D'_{33} = D_{33}, \quad p, q = 1, 2 \tag{2.8}$$

The invariance of Eqs. (1.14), the boundary conditions (1.15) and relations (1.17)–(1.19) under the substitutions (2.4), (2.5) and (2.8) is now immediately obvious, which proves the theorem. □

Remark. The theorem also holds in the case of an orthotropic medium if one of the axes of orthotropy is parallel to the y_3 axis of the rod. Moreover, inhomogeneity of the material is permitted with respect to the coordinates y_1 and y_2 subject to the condition that the explicit dependence of the elastic potential on these coordinates satisfies the requirement

$$W(G_{sk}, y_1, y_2) = W(G_{sk}, -y_1, -y_2)$$

Suppose $u_k = f_k(y_1, y_2)$ ($k = 1, 2, 3$) is the solution of the boundary-value problem (1.14), (1.15), (1.17), (1.18). By virtue of the theorem, the functions

$$u_1 = -f_1(-y_1, -y_2), \quad u_2 = -f_2(-y_1, -y_2), \quad u_3 = f_3(-y_1, -y_2)$$

satisfy the same boundary-value problem. From the uniqueness of the solution, we obtain

$$f_p(-y_1, -y_2) = -f_p(y_1, y_2), \quad f_3(-y_1, -y_2) = f_3(y_1, y_2), \quad p = 1, 2$$

Thus, if the cross-section possesses central symmetry, the solution of the two-dimensional boundary-value problem possesses the property

$$u_p(-y_1, -y_2) = -u_p(y_1, y_2), \quad u_3(-y_1, -y_2) = u_3(y_1, y_2) \tag{2.9}$$

From relations (1.6) and (2.9), we have the equalities

$$X_p(-y_1, -y_2, y_3) = -X_p(y_1, y_2, y_3), \quad p = 1, 2 \tag{2.10}$$

which imply that the cross-section of the horizontal plane of the deformed beam also possesses central symmetry and the X_3 axis, that is, the line $X_1 = X_2 = 0$, passes through the centres of all the cross-sections. In the special case, when the point $y_1 = y_2 = 0$ belongs to the domain σ, that is, the naturally twisted beam does not have a cavity in the central part, it follows from equalities (2.10) that $X_p(0, 0, y_3) = 0$ ($p = 1, 2$). This implies that a material straight line passing through the centres of the cross-sections of the undeformed rod remains a straight line after the stretching compression of the rod and also intersects the horizontal planes at the point $y_1 = y_2 = 0$.

Hence, the longitudinal force F_3, which is the equivalent of the forces which have to be applied to the end of the beam to produce the deformation (1.6), pass through the centre of the cross-section in the case of a beam with a centrally symmetric cross-section.

After solving the two-dimensional boundary-value problem in the cross-section σ, the longitudinal force F_3 and the torque M_3 will be known functions of the parameters ψ and λ. Inverting these functions, we determine the values of the angle of twist $\psi-\alpha$ and the axial elongation $\lambda-1$ using the known values of the longitudinal force and the torque. The energy relations of the non-linear theory of torsion and stretching of a naturally twisted rod

$$F_3(\psi,\lambda)=\frac{\partial\Pi(\psi,\lambda)}{\partial\lambda},\quad M_3(\psi,\lambda)=\frac{\partial\Pi(\psi,\lambda)}{\partial\psi}$$
$$\Pi(\psi,\lambda)=\iint W[u_k(y_1,y_2,\psi,\lambda);\psi,\lambda]d\sigma \tag{2.11}$$

are proved by the method in Ref. 7. Here, Π is a functional of the linear potential energy of an elastic rod calculated on the solution $u_k(y_1, y_2, \psi, \lambda)$ of the two-dimensional boundary-value problem (1.14), (1.15), (1.17), (1.18). According to relations (2.11), the function $\Pi(\psi, \lambda)$ completely defines the deformation properties of the rod in the case of stretching compression and torsion and, in particular, describes the non-linear interaction of the longitudinal and torsional deformations.

In the problem of the three-dimensional bending of a naturally twisted beam, it can be proved, in a similar way to that described earlier in Ref. 7, that the onset of the deformation (1.19) requires the application of a system of forces to the ends of the beam, which is statically equivalent to the resultant of $\mathbf{F}=F_1\mathbf{i}_1$ and the moment $\mathbf{M}=M_1\mathbf{i}_1$. The force $\mathbf{F}$ is applied at a point of the X_1 axis, that is, the axis of the helical surface into which the lateral surface of the beam is transformed after deformation (1.19). The energy relations

$$F_1(\omega,l)=\frac{\partial\Pi(\omega,l)}{\partial l},\quad M_1(\omega,l)=\frac{\partial\Pi(\omega,l)}{\partial\omega}$$

which are similar to (2.11), hold for the magnitude of the force F_1 and the magnitude of the moment M_1, where $\Pi(\omega, l)$ is the linear potential energy of the rod calculated for the solution of the two-dimensional boundary-value problem (1.25), (1.27), (1.15).

3. Compatibility equations and stress functions

It has been assumed above that the functions $u_k(y_1, y_2)$ and $\upsilon_k(y_1, y_2)$ ($k=1, 2, 3$) are the principal unknowns in the two-dimensional boundary-value problems (1.14), (1.15), (1.17), (1.18) and (1.25), (1.27) respectively. Formulations of the problem for a cross-section of a beam with another choice of the unknowns are possible. For example, the components of the gradient of the deformation C_{sk} can be adopted as the principal unknowns, while the functions $u_k(\upsilon_k)$ can be eliminated from the system of equations. To do this, it is necessary to consider the problem of determining the functions $u_k(\upsilon_k)$ in the domain σ from the system of Eqs. (1.9), ((1.22)), assuming that the functions C_{sk} are specified. The necessary and sufficient conditions for these systems to be solvable, which can be referred to as compatibility equations, have the form

$$\begin{aligned}&\psi C_{p2}+(-1)^{p+1}\alpha C_{(3-p)1}+\alpha y_1C_{21,p}-\alpha y_2C_{11,p}+C_{31,p}=0\\&\psi C_{p1}-(-1)^{p+1}\alpha C_{(3-p)2}-\alpha y_1C_{22,p}+\alpha y_2C_{12,p}-C_{32,p}=0;\quad p=1,2\\&C_{13,2}-C_{23,1}=0,\quad C_{33}+\alpha y_1C_{23}-\alpha y_2C_{13}-\lambda=0\end{aligned}\tag{3.1}$$

in the stretching and torsion problem, and

$$\begin{aligned}&\omega C_{p2}-(-1)^{p+1}\alpha C_{(3-p)3}+\alpha y_2C_{13,p}-\alpha y_1C_{23,p}-C_{33,p}=0\\&\omega C_{p3}+(-1)^{p+1}\alpha C_{(3-p)2}-\alpha y_2C_{12,p}+\alpha y_1C_{22,p}+C_{32,p}=0;\quad p=1,2\\&C_{11,2}-C_{21,1}=0,\quad C_{31}-\alpha y_2C_{11}+\alpha y_1C_{21}-l=0\end{aligned}\tag{3.2}$$

in the bending problem.

When conditions (3.1) are satisfied, the functions u_1 and u_2 are uniquely defined by relations (1.9), using the functions C_{sk} which have been specified in the simply connected domain σ, while the function u_3 is defined apart from an arbitrary additive constant. If the domain σ is multiply connected, the function u_3 will, generally speaking, be multiple-valued.

Satisfying conditions (3.2) guarantees the existence of the functions $v_k (k = 1, 2, 3)$ which are determined from relations (1.22). In the case of a multiply connected domain σ, the functions v_2 and v_3 are found uniquely while the function v_1 can be multiple-valued.

The compatibility equations (3.1), together with the equilibrium equations (1.14), in which the stresses D_{sk} are assumed to be expressed in terms of the quantities $C_{ij}(i, j = 1, 2, 3)$ by means of the constitutive relations (1.12) and boundary conditions (1.15), constitute a formulation of a boundary-value problem for the domain σ with the unknown functions $C_{ij}(y_1, y_2)$. The boundary-value problem describes the torsion and stretching compression of a naturally twisted rod. Only the last condition of the two necessary conditions for the solution of (1.17) to be unique now remains in which cosβ has to be expressed in terms of the components of the tensor **C** using relations (1.9) and (1.18).

The system of equations in the components of the gradient of the deformation in the problem of the three-dimensional bending of a beam consists of the compatibility Eq. (3.2), the equilibrium Eq. (1.25) and, also, the second integral of relation (1.27) in which, in accordance with relations (1.22), the quantities C_{pn} must appear instead of the derivatives $v_{n,p}$.

The above-mentioned formulations of the boundary-value problem for a cross-section of a beam with the quantities C_{sk} as the unknowns do not have a variational formulation. This drawback can be removed by adopting the components of the Piola stress tensor D_{rt} as the unknown functions. To do this, it is necessary to express the quantities C_{sk} in terms of the stresses D_{rt} following the procedure which has been explained earlier in Ref. 7. As a result, the compatibility Eqs. (3.1) and (3.2) will be written in terms of the stresses D_{rt}.

It is easily verified that the equilibrium Eq. (1.14) are identically satisfied by the following substitution

$$\begin{aligned}
&D_{11} = \alpha y_2 H_2 + \psi\Phi_{11}, \quad D_{12} = -\alpha y_2 H_1 + \psi\Phi_{12} \ (1 \leftrightarrow 2)\\
&D_{13} = \Phi_{0,2} - \alpha y_2 \Phi_{33}, \quad D_{23} = -\Phi_{0,1} + \alpha y_1 \Phi_{33}, \quad D_{33} = \Phi_{33}\\
&D_{31} = -H_2, \quad D_{32} = H_1; \quad H_p \equiv \Phi_{1p,1} + \Phi_{2p,2}, \quad p = 1, 2
\end{aligned} \tag{3.3}$$

and that the equilibrium equations (1.25) are identically satisfied by the substitution

$$\begin{aligned}
&D_{11} = \chi_{0,2} - \alpha y_2 \chi_{31}, \quad D_{21} = \alpha y_1 \chi_{31} - \chi_{0,1}, \quad D_{31} = \chi_{31}\\
&D_{1p} = \omega\chi_{1p} + (-1)^p \alpha y_2 K_{5-p}, \quad D_{2p} = \omega\chi_{2p} - (-1)^p \alpha y_1 K_{5-p}\\
&D_{32} = -K_3, \quad D_{33} = K_2; \quad K_p \equiv \chi_{1p,1} + \chi_{2p,2}, \quad p = 2, 3
\end{aligned} \tag{3.4}$$

We shall call the six functions $\Phi_0, \Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}, \Phi_{33}$, referring to the problem of the stretching and torsion of a beam, and the six functions $\chi_0, \chi_{12}, \chi_{22}, \chi_{13}, \chi_{23}, \chi_{31}$, referring to the problem of three-dimensional bending, the stress functions. They have to satisfy the compatibility Eqs. (3.1) and (3.2) and the boundary conditions on the contour ∂σ of the cross-section σ respectively. These boundary conditions are derived from relations (1.15), (3.3), (3.4) and have the form

$$n_1\Phi_{11} + n_2\Phi_{21} = 0, \quad n_1\Phi_{12} + n_2\Phi_{22} = 0, \quad \partial\Phi_0/\partial s = 0 \tag{3.5}$$

in the torsion problem and

$$\partial\chi_0/\partial s = 0, \quad n_1\chi_{12} + n_2\chi_{22} = 0, \quad n_1\chi_{13} + n_2\chi_{23} = 0 \tag{3.6}$$

in the bending problem. Here s is the current length of the arc of the boundary contour. If the domain σ is simply connected, the boundary conditions for the functions Φ_0 and χ_0 can be replaced, without loss of generality, in conditions (3.5) and (3.6) by the conditions

$$\Phi_0|_{\partial\sigma} = 0, \quad \chi_0|_{\partial\sigma} = 0$$

The replacement of the unknowns $u_k(y_1, y_2)$, $v_k(y_1, y_2)$ by the stress functions in the two-dimensional boundary-value problem for the domain σ can be characterized as a transformation of a problem of the Neumann type with non-linear boundary conditions into a problem of the Dirichlet type with linear boundary conditions.

The use of stress functions enables us to provide variational formulations of two-dimensional problems for the cross-section of a beam. In particular, the functional of the Castigliano variational principle (the principle of additional energy) in the problem of stretching and torsion is written in the form

$$\Pi[\Phi_0, \Phi_{pq}, \Phi_{33}] = \iint V(\Phi_0, \Phi_{pq}, \Phi_{33})d\sigma, \quad p, q = 1, 2 \tag{3.7}$$

Here V is the specific additional energy of the elastic material[5,8] which is a function of the Piola stress tensor and is related to the specific potential energy of deformation $W(\mathbf{C})$ by a Legendre transformation. It is assumed in equality (3.7) that the components of the Piola tensor are expressed in terms of the stress functions using formulae (3.3). The stress functions which are varied must be differentiable and satisfy boundary conditions (3.5). The compatibility Eq. (3.1) follows from the stationarity of the functional Π.

Other variational principles of the non-linear theory of the torsion and bending of a naturally twisted beam are formulated in a similar way to that proved earlier in Ref. 7.

4. The torsion and stretching of a circular cylinder with helical anisotropy

In the case of a definite type of curvilinear anisotropy and inhomogeneity of the material, a rod of circular cross-section can be regarded, as for as its mechanical properties are concerned, as a naturally twisted body. A cylinder with helical (spiral) anisotropy serves as an example. Problems the deformation of a rod with this type of anisotropy have been considered earlier in Refs. 4,9 within the framework of the linear theory of elasticity.

Consider an elastic solid in the form of a hollow circular cylinder. We will denote the cylindrical coordinates of the points of the solid in the reference configuration by r, φ and z and the unit vectors tangential to the coordinate lines by $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$. We shall assume that the elastic material is orthotropic. The directions of the principal axes of orthotropy at each point of the solid are specified by the orthonormal basis $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$. The orientation of this basis with respect to the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$ is defined by the formulae

$$\begin{aligned}&\mathbf{q}_1 = \mathbf{e}_r, \quad \mathbf{q}_2 = \mathbf{e}_\varphi \cos\tau(r) + \mathbf{e}_z \sin\tau(r)\\&\mathbf{q}_3 = -\mathbf{e}_\varphi \sin\tau(r) + \mathbf{e}_z \cos\tau(r)\end{aligned} \tag{4.1}$$

According to formulae (4.1), the vectors $\mathbf{q}_2$ and $\mathbf{q}_3$ at each point of the solid with the coordinates r, φ and z have the direction of the tangents to the helices located on the cylinder of radius r and to the generatrices of the angles τ and $\pi/2 - \tau$ with the plane z = const. In the general case, the angle τ is assumed to be a differentiable function of the radial coordinate r.

The specific potential energy of deformation of a linearly elastic, orthotropic material W can be represented[10] as a function of the following form

$$W = W(a_1, a_2, a_3, a_{12}, a_{13}, a_{23}, a_0; r) \tag{4.2}$$

$$\begin{aligned}&a_k = \mathbf{q}_k \cdot \mathbf{G} \cdot \mathbf{q}_k, \quad a_{km} = (\mathbf{q}_k \cdot \mathbf{G} \cdot \mathbf{q}_m)^2, \quad k, m = 1, 2, 3, \quad m > k\\&a_0 = \det\mathbf{G}\end{aligned} \tag{4.3}$$

The explicit dependence of the elastic potential on the radial coordinate in (4.2) occurs in the case of radial inhomogeneity of the material. From relations (1.12), (4.2) and (4.3), we find the expression for the Kirchhoff stress tensor

$$\begin{aligned}\frac{1}{2}\mathbf{P} = &\sum_{\substack{k, m = 1\\ m > k}}^{3} \frac{\partial W}{\partial a_{km}} \mathbf{q}_k \cdot \mathbf{G} \cdot \mathbf{q}_m (\mathbf{q}_k \otimes \mathbf{q}_m + \mathbf{q}_m \otimes \mathbf{q}_k) +\\&+ \sum_{k=1}^{3} \frac{\partial W}{\partial a_k} \mathbf{q}_k \otimes \mathbf{q}_k + \frac{\partial W}{\partial a_0} a_0 \mathbf{G}^{-1}\end{aligned} \tag{4.4}$$

In the case of an incompressible material when $a_0 = 1$, the last term in Eq. (4.4) is replaced by the expression $-p\mathbf{G}^{-1}$, where p is the pressure, which is not defined in terms of the deformation and is an unknown function of the coordinates.

We will denote the cylindrical coordinates of the points of the solid after deformation by R, Φ and Z and seek a solution of the problem of the torsion and stretching of a circular cylinder in the form of a special case of the family (1.6)

$$R = R(r), \quad \Phi = \varphi + \psi z, \quad Z = \lambda z \tag{4.5}$$

From relations (4.5), we find a representation of the gradient of the deformation and of the measure of Cauchy deformation

$$\begin{aligned}
\mathbf{C} &= \frac{dR}{dr}\mathbf{e}_r \otimes \mathbf{e}_R + \frac{R}{r}\mathbf{e}_\varphi \otimes \mathbf{e}_\Phi + \psi R\mathbf{e}_z \otimes \mathbf{e}_\Phi + \lambda\mathbf{e}_z \otimes \mathbf{e}_z \\
\mathbf{G} &= \left(\frac{dR}{dr}\right)^2\mathbf{e}_r \otimes \mathbf{e}_r + \frac{R^2}{r^2}\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \frac{\psi R^2}{r}(\mathbf{e}_\varphi \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\varphi) + (\psi^2 R^2 + \lambda^2)\mathbf{e}_z \otimes \mathbf{e}_z \\
\mathbf{G}^{-1} &= \left(\frac{dR}{dr}\right)^{-2}\mathbf{e}_r \otimes \mathbf{e}_r + \left(\frac{r^2}{R^2} + \frac{\psi^2 r^2}{\lambda^2}\right)\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi - \frac{\psi r}{\lambda^2}(\mathbf{e}_\varphi \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\varphi) + \frac{1}{\lambda^2}\mathbf{e}_z \otimes \mathbf{e}_z \\
\mathbf{e}_R &= \mathbf{e}_r\cos(\Phi - \varphi) + \mathbf{e}_\varphi\sin(\Phi - \varphi) \\
\mathbf{e}_\Phi &= -\mathbf{e}_r\sin(\Phi - \varphi) + \mathbf{e}_\varphi\cos(\Phi - \varphi)
\end{aligned} \tag{4.6}$$

The equalities

$$\mathbf{q}_1 \cdot \mathbf{G} \cdot \mathbf{q}_2 = \mathbf{q}_3 \cdot \mathbf{G} \cdot \mathbf{q}_1 = 0$$

follow from relations (4.1) and (4.6), and from these and from formulae (4.4) and (4.6), we obtain

$$\mathbf{q}_1 \cdot \mathbf{P} \cdot \mathbf{q}_2 = \mathbf{q}_3 \cdot \mathbf{P} \cdot \mathbf{q}_1 = 0 \tag{4.7}$$

Taking account of the fact that $\mathbf{D} = \mathbf{P} \cdot \mathbf{C}$ and relations (4.1), (4.6) and (4.7), we find the representation of the Piola stress tensor in the problem of the torsion of a circular cylinder made of a compressible material with helical orthotropy

$$\begin{aligned}
\mathbf{D} &= D_{rR}(r)\mathbf{e}_r \otimes \mathbf{e}_R + D_{\varphi\Phi}(r)\mathbf{e}_\varphi \otimes \mathbf{e}_\Phi + D_{\varphi z}(r)\mathbf{e}_\varphi \otimes \mathbf{e}_z + \\
&+ D_{z\varphi}(r)\mathbf{e}_z \otimes \mathbf{e}_\varphi + D_{zz}(r)\mathbf{e}_z \otimes \mathbf{e}_z
\end{aligned} \tag{4.8}$$

Since the components of the tensor $\mathbf{D}$ in equality (4.8) are expressed using the constitutive relations (4.4) in terms of the function $R(r)$, substitution of expression (4.8) into the equilibrium Eq. (1.11) leads to a linear second-order ordinary differential equation in this function

$$\frac{dD_{rR}}{dr} + \frac{D_{rR} - D_{\varphi\Phi}}{r} - \psi D_{z\Phi} = 0 \tag{4.9}$$

In the case of an incompressible material, the Piola stress tensor is expressed not only in terms of the function $R(r)$, but also in terms of the unknown pressure $p(r, \varphi, z)$:

$$\mathbf{D} = dW/d\mathbf{C} - p\mathbf{C}^{-T}$$

Two of the three equilibrium Eq. (1.11) now reduce to the form

$$\partial p/\partial\varphi = 0, \quad \partial p/\partial z = 0$$

The radial coordinate of the deformed cylinder is determined from the incompressibility condition

$$\det\mathbf{G} = \left(\frac{dR}{dr}\right)^2\frac{\lambda^2 R^2}{r^2} = 1$$

and has the form

$$R = \sqrt{R_1^2 + \lambda^{-1}(r^2 - r_1^2)}, \quad R_1 = R(r_1) \tag{4.10}$$

where r_1 and R_1 are the internal radius of the cylinder before and after deformation respectively. In the case of an incompressible material, Eq. (4.9) serves to determine the pressure function $p(r)$, which is found apart from a single arbitrary constant. This constant, as well as the constant R_1, are determined from the boundary conditions on the internal $r = r_1$ and external $r = r_0$ surfaces of the cylinder

$$D_{rR}\big|_{r = r_1} = D_{rR}\big|_{r = r_0} = 0 \tag{4.11}$$

In the case of a compressible material, the constraints (4.11) serve as boundary conditions for the second-order equation in $R(r)$.

By virtue of representation (4.8), the system of stresses acting in the cross-section $z = \text{const}$ leads to a longitudinal force F_3 and a torque M_3, which are expressed by the formulae

$$F_3(\lambda, \psi) = 2\pi\int_{r_1}^{r_0} D_{zz} r dr, \quad M_3(\lambda, \psi) = 2\pi\int_{r_1}^{r_2} D_{z\varphi} R r dr$$

Hence, the Saint-Venant problem on the torsion and stretching of a non-linearly elastic circular cylinder with helical anisotropy has been reduced to a boundary-value problem for an ordinary differential equation.

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References

1. Lur'ye AI, Dzhanelidze GYu. The Saint-Venant problem for naturally twisted rods. *Dokl Akad Nauk SSSR* 1939;**24**(1):23–6, 3: 226–8; 4: 325–6.
2. Shorr BF. On the theory of twisted non-uniformly heated rods. *Izv Akad Nauk SSSR Otd Tehkn Nauk Mekhanika i Mashinostroyeniye* 1960;**1**:141–51.
3. Berdichevskii VL, Starosel'skii LA. The bending, stretching and torsion of naturally twisted rods. *Prikl Mat Mekh* 1985;**49**(6):978–91.
4. Ustinov YuA. *Saint-Venant Problems for Pseudocylinders*. Moscow: Fizmatlit; 2003.
5. Lur'ye AI. *The Non-Linear Theory of Elasticity*. Moscow: Nauka; 1980.
6. Lur'ye AI. *The Theory of Elasticity*. Moscow: Nauka; 1970.
7. Zubov LM. Large deformation in the three-dimensional bending of prismatic solids. *Prikl Mat Mekh* 2004;**68**(3):507–15.
8. Zubov LM. The principle of stationarity of additional work in the non-linear theory of elasticity. *Prikl Mat Mekh* 1970;**34**(2):241–5.
9. Musalimov VM, Mokryak SYa. Some problems for a helically-anisotropic medium. In: *Continuum Mechanics*. Tomsk; 1983. pp. 88–96.
10. Green AE, Adkins JE. *Large elastic deformations and non-linear continuum mechanics*. Oxford: Clarendon Press; 1960.

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